REMARKS ON DUALITY AND SEMIGROUPS

BY
N. J. ROTHMAN*

ABSTRACT
This paper discusses the duality theory for compact commutative idempotent semigroups and compact commutative inverse semigroups. The major difference from earlier work is the use of semicharacters which are measurable with respect to some measure on the semigroup. Duality theorems are proved using measures which give different dual spaces.

1. Introduction. The purpose of these remarks is a discussion of duality theorems for compact commutative idempotent semigroups and compact commutative inverse semigroups. In [1], Austin considered the problem of finding an analogue of the Pontrjagin duality theorem applicable to topological semigroups. In [2], the Bakers improved on some of Austin's results and introduced the involution concept to obtain results outside the category of inverse semigroups. However, in both [1] and [2] the range of the semigroup homomorphisms was taken as D, the multiplicative semigroup of complex numbers of modulus less than or equal to one, and the homomorphisms were taken to be continuous. In view of [4], [7], [9], [13] and [14], it seems possible to deal with measurable homomorphisms into D, instead of considering a new notion of semicharacters as in [5].

Let S be a compact commutative idempotent semigroup. It is known from [3], that the continuous D-valued homomorphisms, $S^\wedge$, on S, with the compact open topology are such that the D-valued homomorphisms, $S^{\wedge\wedge}$, on $S^\wedge$ (which is discrete [1]) with the compact open topology are such that $S^{\wedge\wedge} = S$ if and only if S is totally disconnected. This means that any component of a compact commutative idempotent semigroup is considered as a point by the continuous

* This was written while the author was a guest of the Hebrew University of Jerusalem, Israel, and was supported in part by the National Science Foundation.
Received July 28, 1969, and in revised form January 26, 1970.

homomorphisms. However, from results in [8] and [11], if S is also connected and linearly ordered by $\leqq$, where for e and f in S, $e \leqq f$ if $ef = e$, then S is isomorphic to $[0,1]$ with minimum multiplication [12] and the Lebesgue measurable D-valued homomorphisms, S^*, identified modulo equal almost everywhere, are isomorphic to $[0,1]$ with maximum multiplication and conversely, so that $S^{**} = S$.

We will consider here those semigroups S which are compact commutative idempotent semigroups and are linearly ordered by $\leqq$. A measure will be introduced on such S so that the measurable D-valued homomorphisms, S^*, can be identified with the maximal ideal space of $L^1(S)$, given the Gelfand topology, and again realized as such an S and finally obtain $S^{**} = S$. The extension of these results to certain compact commutative inverse semigroups [6], will be outlined in the final section.

2. In order to point out the differences between the material in these remarks and earlier results, we first consider an example of a compact, commutative, totally disconnected idempotent semigroup and compute $S^\wedge$ and S^*.

EXAMPLE 1. Let $S = \{0\} \cup \{2^{-n}\}_{n=1}^{\infty}$ with minimum multiplication and the relative topology of the line. From the results of Austin $S^\wedge$ is discrete and easily seen to be isomorphic to S with maximum multiplication and the discrete topology. Since S is: otally disconnected $S^{\wedge\wedge} = S$.

A natural measure to introduce on S is the counting measure (each point having mass 1). The measurable homomorphisms on S are then all D-valued homomorphisms and S^* can be identified as the maximal ideal space of $l^1(S)$ by applying the results of Hewitt and Zukerman [8]. The topology on S^* is that of pointwise convergence, S^* is compact and can easily be seen to be isomorphic to $\{-1\} \cup \{0\} \cup \{2^{-n}\}_{n=1}^{\infty}$ with maximum multiplication and the relative topology of the line. Since S^* is again of the same nature as S, the measure on S^* of assigning mass 1 to each point except $\{0\}$ is such that S^{**} is isomorphic to S. However, if S^* was given the counting measure then $S^{**} \neq S$.

From this example, it is seen that even for a totally disconnected S, the choice of counting measure treats S as discrete and $L^1(S)$ as $l^1(S)$. It follows that points such as $\{0\}$ in S^* above must have mass zero for duality to occur. This leads us to the following.

DEFINITION. For any semigroup S, let E denote the set of idempotent elements of S and let E_g the set of those idempotent elements of S such that $e \in E_g$ if and only if $[f: f \neq e, fe = f]$ is open and closed in E.

THEOREM 1. *Let S be a compact commutative idempotent semigroup which is linearly ordered by $\leqq$ and totally disconnected. If M is counting measure on E_g and zero elsewhere then S^*, the M-measurable, D-valued homomorphisms on S, is a locally compact commutative idempotent semigroup which is linearly ordered, totally disconnected and has a measure M^* (counting measure on $(E^*)_g$ and zero elsewhere), such that S^{**}, the M^*-measurable, D-valued homomorphisms on S^*, is isomorphic to S algebraically and topologically under the evaluation mapping.*

PROOF. Note that E_g is a subsemigroup of S since $ef = f$ or e for any e and f in S. When M is chosen as the counting measure on E_g and zero elsewhere, then $L^1(S, M)$ is $l^1(E_g)$ on S^* can be identified as the maximal ideal space of $E^1(E_g)$ as in [7]. Since all D-valued homomorphisms on S take only the values 0 and 1, S^* is seen to be totally disconnected. The linear order on S carries over to a linear order on S^* since each homomorphism is the characteristic function of an order interval $[e, 1]$ (note that S must possess an identity element 1). When M^* is taken as counting measure on $(E^*)_g$, S^{**} is then identified with the maximal ideal space of $l^1((E^*)_g)$. The topology on S^* and S^{**} is clearly the topology of pointwise convergence as function spaces.

Let us now consider the evaluation mapping of $S \to S^{**}$ given by $x \to \bar{x}$ where $\bar{x}(\tau) = \tau(x)$ for $\tau \in S^*$. We show that this mapping is an algebraic isomorphism and a homeomorphism. Since S is totally disconnected, the continuous homomorphisms separate points and the mapping is one-to-one and a homomorphism so an isomorphism.

To see that the mapping in onto, let $\theta \in S^{**}$ and let $\tau_0 = \sup[\tau \in S^*: \theta(\tau) = 0]$. We assume without loss of generality that $\theta \not\equiv 1$. Now, $\theta(\tau_0) = 0$ since if not [illegible] a net $\{\tau_\alpha\} \subset (E^*)_g$ with $\tau_\alpha \uparrow \tau_0$ and thus $M^*(\{\tau_0\}) = 0$ so θ is equivalent to θ' with $\theta'(\tau_0) = 0$ and $\theta' = \theta$ elsewhere. Let $x_0 = \inf[x \in S: \tau_0(x) = 1]$. We show that $\bar{x}_0 = \theta$ a.e. If $\bar{x}_0(\tau_0) = 0$, then for $\tau > \tau_0$, $\bar{x}_0(\tau) = 1 = \theta(\tau)$ and for $\tau < \tau_0$, $\bar{x}_0(\tau) = \bar{x}_0(\tau)\bar{x}_0(\tau_0) = 0 = \theta(\tau)$. If $\bar{x}_0(\tau_0) = 1$, then $x \in E_g$ and again $\bar{x}_0 = \theta$. So the mapping is onto.

Since S is compact, we need only show the continuity of the mapping to obtai n a homeomorphism. Let $\{x_\alpha\}$ be a net in S with $x_\alpha \to x$. If $x \in E_g$ then there is an α_0 so that $\alpha > \alpha_0$ implies $x_\alpha x = x$. Thus for any $\tau \in S^*$ with $\tau(x) = 1$, $\tau(x_\alpha) = 1$

and $\bar{x}_\alpha(\tau) \to \bar{x}(\tau)$. For any $\tau \in S^*$ with $\tau(x) = 0$, there is a $y \in E_g$ such that $yx = x$, $\tau(y) = 1$ and either for all z with $yz = z \neq y$, $\tau(z) = 0$ so that $\tau(x_\alpha) = 0$ for all α with $x_\alpha y = x_\alpha \neq y$ and $\bar{x}_\alpha(\tau) \to \bar{x}(\tau)$ or $\tau(y) = 1$ for all y with $xy = x$, and $y \neq x$. Then τ is a point of measure 0 in S^* and $\tau(x)$ may be taken to have value 1 and the preceding gives $\bar{x}_\alpha(\tau) \to \bar{x}(\tau)$. If $x \notin E_g$ and $\tau \in S^*$ with $\tau(x) = 1$, then either there is a $y \in S$, $y < x$ and $\tau(y) = 1$ and hence $\bar{x}_\alpha(\tau) \to \bar{x}(\tau)$ or for all $y < x$, $\tau(y) = 0$ and hence $\tau(x)$ can be taken as zero also. Thus, if $\tau(x) = 0$ either there is a $y > x$ so that $\tau(y) = 0$ and $\bar{x}_\alpha(\tau) \to \bar{x}(\tau)$ or $\tau(y) = 1$ for all $y > x$. Thus τ is that homomorphism such that $\tau(y) = 0$, for $y < x$, and $\tau(y) = 1$, for $y > x$ and $\tau(x) = 1$ or 1. Thus $\bar{x}_\alpha(\tau) \to \bar{x}(\tau)$ and the mapping is continuous and S and S^{**} are homeomorphic.

Let S be a compact commutative totally disconnected idempotent semigroup linearly ordered by $\leqq$. There is a "natural" duality for S and S^* which is different from that of Theorem 1. That is declaring S^* to be the order dual of S. In order to clarify the precise role that the measures on S and S^* must play for this duality, the following necessary and sufficient condition is evident.

LEMMA. *There is a measure m on S (m^* on S^*) such that:* (a) *if $e,f \in S$ with $e \neq f$ then $\chi_{[e,1]}$ and $\chi_{[f,1]}$ are not equal almost everywhere and* (b) *each semicharacter is equal almost everywhere to some $\chi_{[e,1]}$.*

The natural isomorphism of S^{**} and S occurs here when S^* is given the order topology and is the order dual of S with maximum multiplication instead of minimum.†

3. To extend Theorem 1 to the non totally disconnected S, we again first consider an example.

EXAMPLE 2. Let

$$S = \{0\} \cup \{2^{-n}\}_{n=1}^{\infty} \cup [1,2] \cup \{2 + 2^{-n}\}_{n=1}^{\infty}$$

with minimum multiplication and the relative topology of the line. Here, $E_g = S \backslash (1,2]$, but the element 1 in E_g is not to have mass 1 because it has the property that the two homomorphisms $\chi_{(1,2]} \cup \{2 + 2 - n\}_{n=1}^{\infty}$ and $\chi_{[1,2]} \cup \{2 + 2 - n\}_{n=1}^{\infty}$ (χ_A is the characteristic function of A) differ only at 1 and we wish to have them identified, for otherwise S^* would be such that $S^{**} \neq S$.

Let M be that measure on S which is counting measure on $E_g \backslash \{1\}$ and Lebesgue

† The author wishes to thank the referee for the contribution of this idea to this paper.

measure on $[1,2]$. It is not difficult to see that S^* can be thought of as $\{-1\} \cup \{0\} \cup \{2^{-n}\}_{n=1}^{\infty} \cup [1,2] \cup \{3\} \cup \{3+2^{-n}\}_{n=1}^{\infty}$ with maximum multiplication and the relative topology of the line. Since S^* is again similar to S the measure M^* of counting measure on $E_g^*/(\{2\} \cup \{0\} \cup \{3\})$ and Lebesgue measure on $[1,2]$ is such that S^{**} is isomorphic to S. Note that unless care is taken duality can fail when counting measure is on all of E_g.

THEOREM 2. *Let S be a commutative idempotent semigroup which is linearly ordered by $\leqq$ and is compact in the order topology. There is a measure M on S such that S^*, the M-measurable, D-valued homomorphism on S, is a commutative idempotent semigroup which is linearly ordered and locally compact in the order topology. Further, there is a measure M^* on S^* such that S^{**}, the M^*-measurable, D-valued homomorphisms on S^*, is linearly ordered and to S algebraically and topologically, with the evaluation mapping, when S^{**} has the order topology.*

Before proceedings to the proof of Theorem 2, we need the

LEMMA. *Let S be as in the above theorem, then S^* can be obtained from the order dual of S by the deletion and addition of points in the following manner.*

1. *Each element with no immediate predecessor in S but with an immediate successor in S which is not the right hand end point of a nondegenerate component of S is to be deleted, and*
2. *Each element with no immediate successor in S which is not interior to, or the left hand end point of a non-degenerate component of S is to be given an immediate predecessor in S^*.*

PROOF. Let E_1 be the union of all non-degenerate componnets of S and let $E_0 = E_g \backslash E_1$. Let M be that measure on S which assigns mass 1 to each point of E_0 and Lebesgue measure to each non-degenerate component of S. Now the homomorphisms of S are all characteristic functions of sets of the form $[x : x > e]$ or $[x : x \geqq e]$. If $e \in S \backslash E_1$ such that e has no immediate predecessor but has an immediate successor, the characteristic functions of $[x : x \geqq e]$ and $[x : x > e]$ are equal almost everywhere. If $e \in S \backslash E_1$ and e has no immediate successor, then the characteristic functions of $[x : x \geqq e]$ and $[x : x > e]$ are equal almost everywhere if e has no immediate predecessor and are distinct if e has an immediate predecessor. The description of S^* as the order dual of S together with certain additions and deletions as described follows.

PROOF OF THEOREM 2. The introduction of the measures M and M^* on S and

S^* respectively derives from Example 2. Let E_1 be the union of all non-degenerate components of S and let $E_0 = E_g \backslash E_1$. Let M be that measure on S which is counting measure on E_0 and Lebesgue measure on each component of E_1. The measure M^* on S^* is defined analogously as counting measure on $(E^*)_0$ and Lebesgue measure on each component of $(E^*)_1$. Since S and S^* have identity elements, it is clear that S^* and S^{**} are semigroups which are linearly ordered by $\leqq$. By using the structure of S and S^*, it is immediate that S and S^{**} are order isomorphic using the evaluation mapping and hence algebraically and topologically isomorphic.

In order to supply the connecting link between Theorem 1 and 2, we now show that the order topology on S^* is the Gelfand topology as induced by identifying S^* as the maximal ideal space of $L^1(S,M)$. For any $\tau \in S^*$, let x_0 $\inf[x\colon \tau(x) = 1]$, then $\tau = \chi_{S/Sx_0} \cup \{x_0\}$ a.e. M. Thus for $\mu \in L^1(S,M)$, $\mu(\tau) = \int_{x_0}^1 \backslash d\mu$, and since μ is a finite on S μ is continuous in the order topology of S^*. Further, it is clear that the functions μ separate points of S^*, do not all vanish at any point and vanish at infinity (the zero horomorphism) if S^* is not compact. Thus from [8], the Gelfand topology and the order topology are the same. Here we have

PROPOSITION. *Let S be as in Theorem 2, then the Gelfand topology on S^* as the maximal ideal space of $L^1(S,M)$ agrees with the order topology on S^*.*

Let S be a commutative idempotent semigroup which is linearly ordered by $\leqq$ and compact in the order topology. The "natural" duality mentioned in the remarks following Theorem 1 again apply in this situation. Here S^* is the order dual of S with the order topology and with maximum multiplication and the measures are to be taken so that (a) *for e and $f \in S(S^*)$, $\chi_{[e,1]}$ and $\chi_{[f,1]}$ are not equal almost everywhere and* (b) *each semicharacter on $S(S^*)$ is equal almost everywhere to some $\chi_{[e,1]}$.*

It follows from Theorem 2 that there can be more than one measure on S so that a duality theorem occurs.

4. An extension of the preceding results to certain compact commutative inverse semigroups can easily be obtained. Let S be a compact commutative inverse semigroup such that the idempotent elements of S are linearly ordered and such that for each non-degenerate component of E the conditions of [15-theorem 3] are satisfied. It is readily seen that combining techniques here, and in [15] for introducing a measure on S, the results of this paper can also be of-

tained for such semigroups. In particular, the measure M may be chosen as in the proof of Theorem 2 to depend on the idempotent elements. For each $e \in E_0$, M is taken as Haar measure on the maximal group at e and on each component of E_1, M is the measure as chosen in [15]. It is also clear that a "natural" duality can be obtained following the outline of remarks following Theorem 2 for the measure on E and again applying the techniques of [15].

REFERENCES

1. C. W. Austin, *Duality theorems for commutative semigroups*, Trans. Amer. Math. Soc., **109** (1963), 245–256.
2. A. C. Baker and J. W. Baker, *Duality of topological semigroups with involution*, J. London Math. Soc., **44** (1969), 251–260.
3. J. W. Baker and N. J. Rothman, *Separating points by semicharacters in topological semigroups*, Proc. Amer. Math. Soc., **21** (1969), 235–239.
4. J. G. Bergman and N. J. Rothman, *An L^1 algebra for algebraically irreducible semigroups*, Studia Math., **33** (1969), 257–272.
5. D. R. Brown and M. Friedberg, *A new notion of semicharacters*, Trans. Amer. Math. Soc., **141** (1969), 387–401.
6. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1 (Amer. Math. Soc., Providence, R.I.) 1961.
7. E. Hewitt and H. S. Zuckerman, *The l^1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc., **83** (1956), 70–97.
8. E. Hewitt and H. S. Zuckerman, *Structure theory for a class of convolution algebras*, Pacific J. Math., **7** (1957), 913–941.
9. L. J. Lardy, *$L^1(a,b)$ with order convolution*, Studia Math., **27** (1966), 1–8.
10. L. Loomis, *Abstract Harmonic Analysis*, (Van Nostrand Co., New York, N.Y.) 1953.
11. K. A. Ross, *The structure of certain measure algebras*, Pacific J. Math., **11** (1961), 723–737.
12. N. J. Rothman, *Linearly quasi-ordered compact semigroups*, Proc. Amer. Math. Soc., **13** (1962), 352–357.
13. N. J. Rothman, *An L^1 algebra for certain locally compact topological semigroups*, Pacific J. Math., 23 (1967), 143–151.
14. N. J. Rothman, *An L^1 algebra for linearly quasi-ordered compact semigroups*, Pacific J. Math., **23** (1968), 579–588.
15. N. J. Rothman, *Duality and linearly quasi-ordered compact semigroups*, (submitted to J. London Math. Soc.).

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS